HEAT CONDUCTION IN A ROD OF VARIABLE LENGTH DRAWN FROM A MELT WITH CONSTANT VELOCITY

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An approximate solution of the one-dimensional heat-conduction problem for a rod of variable length is obtained by the integral method. The result of calculating the temperature in a rod from the obtained formula is given.

In obtaining crystals from the melt by the Czochralski method a rod (seed) of the same material as the crystal being grown is lowered into the crucible containing the melt. The crucible is heated, the rod is gradually raised, and the crystal is drawn from the melt. We assume that the diameters of the rod and crystal are the same, and the temperature distribution in them is uniform (Fig. 1). We obtain a solution of the problem separately for two periods. In the first period the stationary rod is heated. The second period begins when the rod starts to rise.

We formulate the problem for the first period as follows:

$$\frac{\partial t}{\partial \tau} = a\frac{\partial^2 t}{\partial x^2} - b(t - t_l), \tag{1}$$

$$t(x, 0) = t_l, \quad 0 \leqslant x \leqslant l, \tag{2}$$

$$t(0, \tau) = t_L, \tag{3}$$

$$\frac{\partial t(l, \tau)}{\partial x} = -\frac{\alpha_e}{\lambda}(t - t_l). \tag{4}$$

Here $b = \alpha u/fc\rho$, and the coordinate origin is situated at the bottom end of the rod. Introducing the substitution

$$u = (t - t_l)\exp b\tau, \tag{5}$$

we bring system (1)-(4) to the following form:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial x^2}, \tag{6}$$

$$u(x, 0) = 0, \quad 0 \leqslant x \leqslant l, \tag{7}$$

$$u(0, \tau) = A\exp b\tau, \tag{8}$$

$$\frac{\partial u(l, \tau)}{\partial x} = -\frac{\alpha_e}{\lambda}u(l, \tau), \tag{9}$$

where $A = t_L - t_l$.

To solve the problem we use the integral method, which has recently been applied to the solution of heat-conduction problems [1, 2].

We now introduce the penetration depth $\delta(\tau)$, i.e., the thickness of the layer within which the temperature changes. Then, for the period of time less than τ_0, when $\delta < l$, $u(l, \tau)$ in condition (9) is equal to zero and the problem reduces to the problem for a semiinfinite rod. We will seek the temperature distribution within $\delta(\tau)$ in the form

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$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3. \tag{10}$$

To find the coefficients contained in (10), we assign the following additional boundary conditions to (8)

$$u(\delta, \tau) = 0, \tag{11}$$

$$\frac{\partial u(\delta, \tau)}{\partial x} = 0, \tag{12}$$

$$\frac{\partial^2 u(\delta, \tau)}{\partial x^2} = 0. \tag{13}$$

Using conditions (8), (11), (12), and (13), we obtain

$$u = A \exp b\tau \left(1 - \frac{x}{\delta}\right)^3. \tag{14}$$

To find $\delta(\tau)$ we use the heat balance integral, for which we multiply equation (6) by dx and integrate in the limits $x = 0$ to $x = \delta$. This gives us

$$\frac{d\theta}{d\tau} = a\left[\frac{\partial u(\delta, \tau)}{\partial x} - \frac{\partial u(0, \tau)}{\partial x}\right], \tag{15}$$

where

$$\theta = \int_0^\delta u dx.$$

After simple algebra we obtain an ordinary differential equation for δ

$$\delta \frac{d\delta}{d\tau} + b\delta^2 - 12a = 0 \tag{16}$$

with the initial condition

$$\delta = 0 \quad \text{at} \quad \tau = 0. \tag{17}$$

The solution of the equation has the following form:

$$\delta = 2\sqrt{\frac{3a}{b}}\,[1 - \exp(-2b\tau)]^{1/2}. \tag{18}$$

Putting $\delta = l$ and solving expression (18) for τ, we find the time of heating of the rod

$$\tau_0 = -\frac{1.15}{b}\lg\left(1 - \frac{l^2 b}{12a}\right). \tag{19}$$

At time $\tau = \tau_0$ the rod begins to ascend with constant velocity. The problem for the second period $\tau > \tau_0$ is formulated as follows:

$$\frac{\partial t}{\partial \tau} = a\frac{\partial^2 t}{\partial x^2} - w\frac{\partial t}{\partial x} - b(t - t_l), \tag{20}$$

$$t(x, \tau_0) = f(x), \qquad 0 \leqslant x \leqslant l, \tag{21}$$

$$t(0, \tau) = t_L, \tag{22}$$

$$\frac{\partial t(s, \tau)}{\partial x} = -\frac{\alpha_e}{\lambda}(t - t_l). \tag{23}$$

Condition (23) is imposed on the moving boundary s. The law of motion of the boundary is given by the expression

$$s = l + w(\tau - \tau_0). \tag{24}$$

The coordinate origin is situated on the moving crystallization front, which is assumed to be plane. The velocity of crystallization ω is equal to the sum of the ascent velocity of the rod and the velocity of descent of the melt in the crucible. Since the crystal interacts thermally with the melt, the following conditions will be satisfied during growth of the crystal:

$$t(0, \tau) = t_m(0, \tau) = t_L, \tag{25}$$

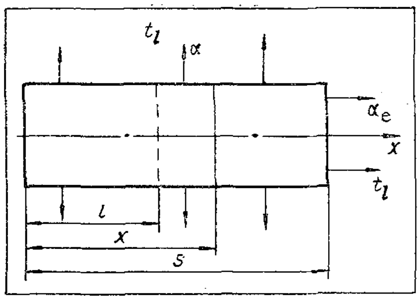

Fig. 1

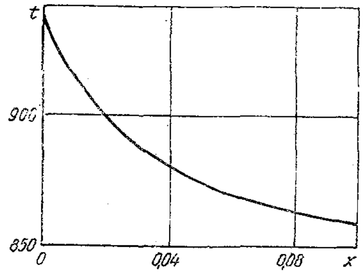

Fig. 2

Fig. 1. Diagram illustrating problem.

Fig. 2. Temperature distribution in rod for s = 0.1 m (x, m; t, °C).

$$\lambda\left|\frac{\partial t(0,\ \tau)}{\partial x}\right| - \lambda_m\left|\frac{\partial t_m(0,\ \tau)}{\partial x}\right| = \rho w L. \tag{26}$$

Hence, the solution of problem (20)-(24) with the adopted values of d, α, α_e, t_l and the thermal parameters of the material has sense from a physical viewpoint for values of ω which satisfy condition (26) for $|\partial t_m(0, \tau)/\partial x|$ varying within fixed limits. This variation can be achieved by varying the power supply from the heater to the melt. Differentiating the obtained solution of the problem, putting x = 0, and substituting it in (26) we can find $|\partial t_m(0, \tau)/\partial x|$, i.e., determine the law of power supply to the melt.

Introducing the substitution

$$u = (t - t_l)\exp[-(\mu x - k\tau)], \tag{27}$$

we bring system (20)-(23) to the following form:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial x^2}, \tag{28}$$

$$u(x,\ \tau_0) = f_1(x), \quad 0 \leqslant x \leqslant l, \tag{29}$$

$$u(0,\ \tau) = A\exp k\tau, \quad \tau > \tau_0, \tag{30}$$

$$\frac{\partial u(s,\ \tau)}{\partial x} = -Fu(s,\ \tau), \tag{31}$$

where

$$\mu = \frac{w}{2a}, \quad k = b + \frac{w^2}{4a}, \quad F = \mu + \frac{\alpha_e}{\lambda}.$$

The initial temperature distribution $f_1(x)$ is obtained from a solution of the first part of the problem for the stationary rod at $\tau = \tau_0$.

We solve the problem by the integral method. We seek the solution in the form of polynomial (10).

We obtain an additional boundary condition

$$\frac{\partial^2 u(0,\ \tau)}{\partial x^2} = \frac{Ak}{a}\exp k\tau \tag{32}$$

by differentiating (30) with respect to τ and substituting in Eq. (28). Conditions (30) and (32) give

$$a_0 = A\exp k\tau, \quad a_2 = \frac{Ak}{2a}\exp k\tau.$$

Using the boundary condition (31), we express the coefficient a_1 in terms of a_3 and write expression (10) as:

$$u = A\exp k\tau\,[1 - m_1 x + m_2 x^2] - a_3(m_3 x - x^3), \tag{33}$$

where

$$m_1 = \frac{F + \frac{k}{a}s + \frac{Fk}{2a}s^2}{1 + Fs}, \quad m_2 = \frac{k}{2a},$$

$$m_3 = \frac{3s^2 + Fs^3}{1 + Fs}.$$

To determine the coefficient a_3 we use the heat balance integral, for which we put u from (33) into (28), multiply by dx, and integrate between the limits 0 and s:

$$\frac{d\theta}{d\tau} = a\left[\frac{\partial u(s, \tau)}{\partial x} - \frac{\partial u(0, \tau)}{\partial x}\right], \tag{34}$$

where

$$\theta = \int_0^s u\,dx = AB_1 \exp k\tau - a_3 B_2, \tag{35}$$

$$B_1 = \frac{12as + 6aFs^2 - 4ks^3 - Fks^4}{12a(1 + Fs)}, \tag{36}$$

$$B_2 = \frac{5s^4 + Fs^5}{4(1 + Fs)}. \tag{37}$$

To find $d\theta/d\tau$ we must bear in mind that B_1 and B_2 are functions of τ, since s is given by expression (24)

$$\frac{d\theta}{d\tau} = A \exp k\tau \frac{dB_1}{d\tau} + Ak \exp k\tau B_1 - \frac{da_3}{d\tau} B_2 - a_3 \frac{dB_2}{d\tau}. \tag{38}$$

Substituting (38) in (34) and using relationship (33), we obtain

$$\frac{da_3}{d\tau} + C(\tau)\, a_3 + D(\tau) = 0, \tag{39}$$

where

$$C(\tau) = \frac{3as^2 + \frac{dB_2}{d\tau}}{B_2},$$

$$D(\tau) = A\, \frac{ks - \frac{dB_1}{d\tau} kB_1}{B_2} \exp k\tau.$$

Differentiating expressions (36) and (37) with respect to τ and using the obtained results, we rewrite the formulas for $C(\tau)$ and $D(\tau)$ in the following form:

$$C(\tau) = \frac{4\left[3a(1 + Fs)^2 + (5s + 5Fs^2 + F^2s^3)\frac{ds}{d\tau}\right]}{(1 + Fs)(5s^2 + Fs^3)}, \tag{40}$$

$$D(\tau) = 4A\varphi(\tau) \exp k\tau, \tag{41}$$

where

$$\varphi(\tau) = \frac{ks(1 + Fs)^2 - \left[1 + Fs - \frac{k}{a}s^2 + \frac{F^2s^2}{2} - \frac{kF}{a}s^3 - \frac{F^2k}{4a}s^4\right]\frac{ds}{d\tau}}{(1 + Fs)(Fs^3 + 5s^4)} - \frac{k\left(s + \frac{Fs^2}{2} - \frac{k}{3a}s^3 - \frac{Fk}{12a}s^4\right)}{Fs^3 + 5s^4}.$$

To find a_3 we solve Eq. (39). We put the solution in the form

$$a_3 = \exp\left[-\int_{\tau_0}^{\tau} C(\tau)\, d\tau\right]\left\{a_3^0 - \int_{\tau_0}^{\tau} D(\tau) \exp\left[\int_{\tau_0}^{\tau} C(\tau)\, d\tau\right] d\tau\right\}. \tag{42}$$

We substitute expression (42) in (33)

$$u = A(1 - m_1 x + m_2 x^2)\exp k\tau$$

$$- \exp\left[-\int_{\tau_0}^{\tau} C(\tau)\,d\tau\right]\left\{\frac{A(1 - m_1^0 l + m_2^0 l^2)\exp k\tau_0}{m_3^0 l - l^3} - \int_{\tau_0}^{\tau} D(\tau)\exp\left[\int_{\tau_0}^{\tau} C(\tau)\,d\tau\right]d\tau\right\}(m_3 x - x^3). \tag{43}$$

To find a_3^0 we used the condition

$$u(l, \tau_0) = 0.$$

To convert to variable t we use the relationship

$$t = t_l + u\exp(\mu x - k\tau), \tag{44}$$

$$t = t_l + A(1 - m_1 x + m_2 x^2)\exp\mu x$$

$$- \exp\left[\mu x - k\tau - \int_{\tau_0}^{\tau} C(\tau)\,d\tau\right]\left\{\frac{A(1 - m_1^0 l + m_2^0 l^2)\exp k\tau_0}{m_3^0 l + l^3} - \int_{\tau_0}^{\tau} D(\tau)\exp\left[\int_{\tau_0}^{\tau} C(\tau)\,d\tau\right]d\tau\right\}(m_3 x - x^3). \tag{45}$$

Expression (45) is the solution of the posed problem. The value of τ_0 is determined from (19). To simplify the calculations we denote the inside variable of integration by ψ and rewrite formula (45) as:

$$t = t_l + A(1 - m_1 x + m_2 x^2)\exp\mu x$$

$$- a_3^0 Z(x)\exp\left[-k\tau - \int_{\tau_0}^{T} C(\psi)\,d\psi\right] + 4AZ(x)\int_{\tau_0}^{\tau} Y(\psi)\,d\psi, \tag{46}$$

where

$$Z(x) = (m_3 x - x^3)\exp\mu x,$$

$$Y(\psi) = \varphi(\psi)\exp\left[-k(\tau - \psi) - \int_{\psi}^{\tau} C(\psi)\,d\psi\right].$$

The integral $M = \int_{\tau_0}^{\tau} Y(\psi)d\psi$ can be determined by numerical methods.

Figure 2 shows the solution of the problem for the following initial data: $\omega = 0.417 \cdot 10^{-4}$ m/sec (2.5 mm/min), $d_{cr} = 0.036$ m, $t_L = 936°C$, $t_l = 500°C$, $\lambda = 17.3$ W/m · deg, $a = 0.0852 \cdot 10^{-4}$ m²/sec, $\alpha = 26.0$ W/m² · deg, $\alpha_e = 50$ W/m² · deg, and $l = 0.02$ m for a crystal 0.1 m long.

Formula (45) becomes the solution for a finite rod if we put $\omega = 0$, $s = l = \text{const}$. The solution will then have the form

$$t = t_L - A(m_1 x - m_2 x^2) - P(m_3 x - x^3)\exp[-(b + C_1)(\tau - \tau_0)] + \frac{D_1}{C_1 + b}(m_3 x - x^3), \tag{47}$$

where

$$C_1 = \frac{12a(1 + Fl)}{5l^2 + Fl^3};$$

$$D_1 = \frac{Ab}{3a}\,\frac{6aF + 4bl + Fbl^2}{5l^2 + Fl^3}:$$

$$P = \frac{A(1 - m_1 l + m_2 l^2)}{m_3 l - l^3} + \frac{D_1}{C_1 + b},$$

m_1, m_2, and m_3 are determined from the formulas given above with s replaced by l.

The solution (47) can be obtained directly if the heat balance integral (34) is written with a constant upper limit. In this case m_1, m_2, m_3, B_1, and B_2 will be constant. When $\tau = \infty$ solution (47) becomes the solution of the corresponding steady-state problem.

The discussed method can be used to obtain approximate solutions for other moving-boundary one-dimensional problems for which exact solutions cannot be obtained.

NOTATION

t is the temperature;
a is the thermal diffusivity;
α is the heat-transfer coefficient;
λ is the thermal conductivity;
u is the perimeter;
ρ is the density;
c is the specific heat;
f is the cross-sectional area;
δ is the penetration depth;
s is the position of moving boundary;
w is the crystallization velocity;
L is the heat of crystallization;
τ is the time.

Subscripts

l denotes the liquid;
cr denotes the crystallization;
e denotes the end;
m denotes the melt.

LITERATURE CITED

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